Pumping, With or Without Choice

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Regularity $\Rightarrow$ Rabin-Scott

Rabin-Scott's Pumping Lemma (Rabin, Scott 1959)

If a language L is regular, then there exists a k s.t. $\forall w \in L.\ |w| \geq k \ \Rightarrow\ \exists x, y, z \in \Sigma^*. \quad w = xyz\ \wedge |xy| \leq k \wedge\ y \neq \epsilon\ \wedge\ \forall n \in \mathbb{N}.\ xy^nz \subseteq L.$

- Non-regular languages satisfy it (Sommerhalder 1980)
- Modus tollens form to show irregularity, e.g. $L = 0^n1^n$

Regularity $\Leftrightarrow$ Jaffe

Jaffe's Pumping Lemma (Jaffe 1978)

A language L is regular iff there exists a k s.t.
$\forall w \in \Sigma^*.\ |w| = k \ \Rightarrow\ \exists x, y, z \in \Sigma^*.\quad w = xyz\ \wedge\ y \neq \epsilon\ \wedge\ \forall u \in \Sigma^*.\ n \in \mathbb{N},\ xyzu \in L\ \Leftrightarrow\ xy^n zu \in L.$

Regularity $\Leftrightarrow$ Jaffe

Definition (Derivative)

$L_x = \{y \in \Sigma^* : xy \in L\}$ is the derivative of L with respect to x.

- e.g. $L = 0^*1^*2^*$, L's derivatives are $L_0 = 0^*1^*2^*$, $L_{01} = 1^*2^*$, $L_{012} = 2^*$.
- Jaffe states that every derivative of language L is equivalent to some derivative with label length shorter than or equal to k.

Myhill-Nerode Theorem (Myhill, Nerode 1958)

A language L is regular iff it has finitely many derivatives.

Regularity $\Leftrightarrow$ EPR

The block pumping (cancellation) property (EPR, 1981)

L $\subseteq \Sigma^*$ has the block pumping property iff there exists a k such that for all w $\in \Sigma^*$ and all ways of inserting k breakpoints into the word, there exist two breakpoints such that the word part in between them can be *repeated* (*omitted*) without affecting word membership.

Theorem of Ehrenfeucht, Parikh and Rozenberg (EPR, 1981)

Regularity, the block pumping property, and the block cancellation property are equivalent.

Picture view

012101010210101

Picture view

0 | 1 2 1 | 0 1 | 0 | 1 0 2 | 1 0 | 1 0 | 1 |

Picture view

$$0 \mid 1\,2\,1 \mid 0\,1 \mid 0 \mid \underbrace{1\,0\,2} \mid 1\,0 \mid 1\,0 \mid 1 \mid$$

Picture view

0121010**102**10101

0121010**102102102...102**10101

012101010101

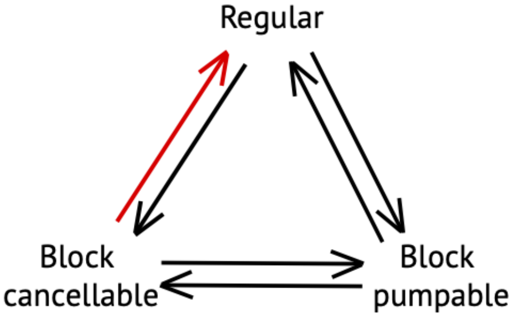
Contributions

- Coq formalization of Jaffe's and EPR's pumping lemmas, and block pumpable language closure properties;
- Clarification of a gap in EPR's proof that block cancelable languages are regular that implicitly uses the Axiom of Choice;
- New choice-free proof using an explicit construction of block cancelable languages from well-formed input sets.

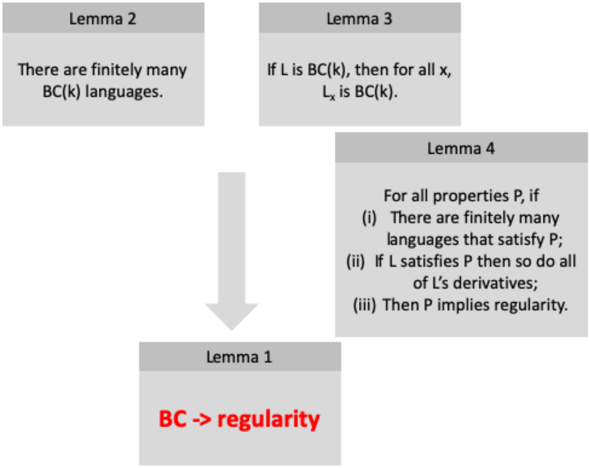
Roadmap

1. Pumping
2. With choice
   - Proof sketch
   - Ramsey's theory
3. Without choice
   - "unshear" function
   - Proof of correctness

EPR's Theorem



EPR's proof, in pictures



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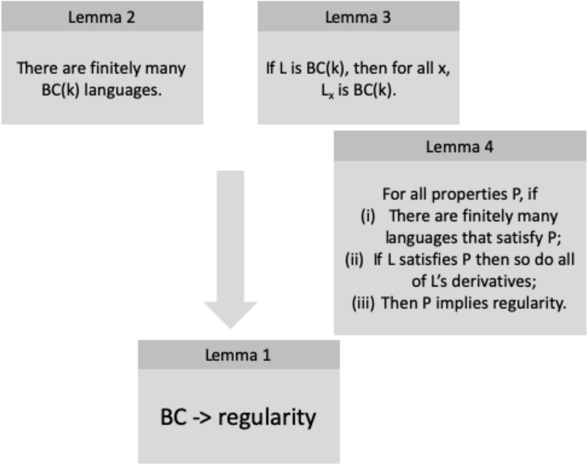

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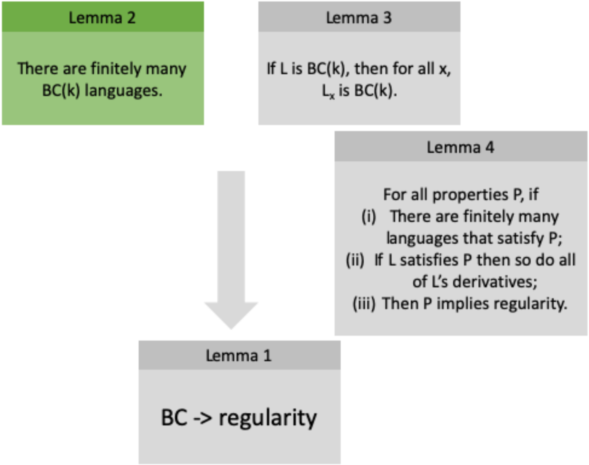
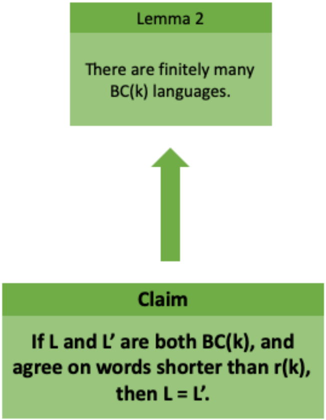
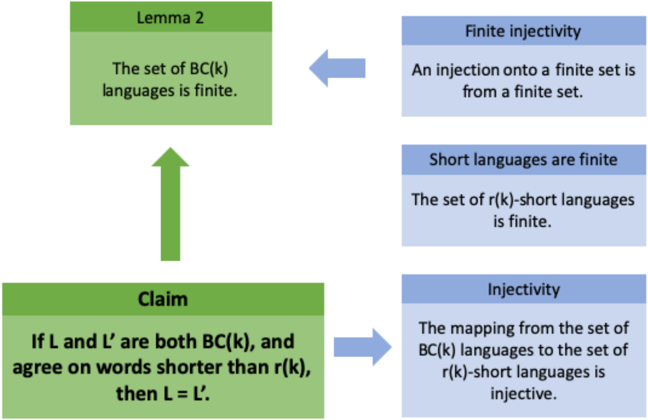


"It is sufficient to show that..."



"It is sufficient to show that..."



Finite injectivity, in Coq

Theorem (Finite injectivity)

```
: forall (P Q : X -> Prop) (f : {x | P x} -> {x | Q x}),
inhabited {x : X | P x} ->
injective P Q f ->
is_finite_dep Q ->
is_finite_dep P.
```

where:

- `X` - language
- `P` - BC(k)
- `Q` - r(k)-short
- `f` - mapping from BC(k) languages to r(k) short languages

Finitehood, in Coq

Definition (Dependent finitehood)

```
is_finite_dep {X : Type} (P : X -> Prop) : Prop :=
exists (L : list {x | P x}), forall (x : {x | P x}), In x L.
```

Definition (Finitehood)

```
is_finite {X : Type} (P : X -> Prop) :=
exists (L : list X), forall (x : X), In x L <-> P x.
```

Axiom of functional choice

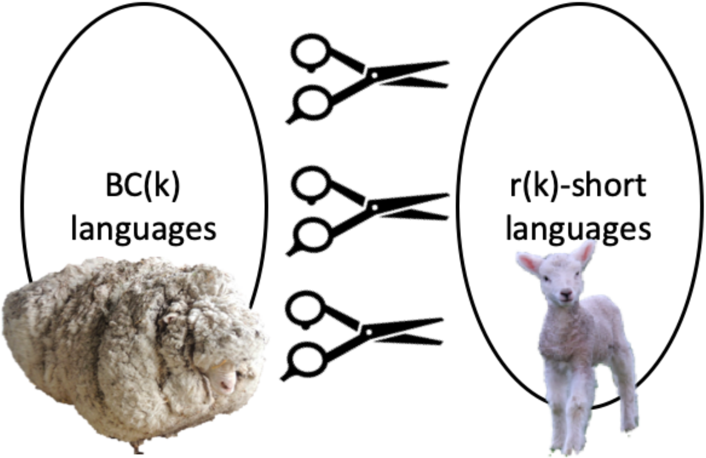
Definition (Functional choice)

```
forall R : A -> B -> Prop,
(forall x : A, exists y : B, R x y) ->
(exists f : A -> B, forall x : A, R x (f x)).
```

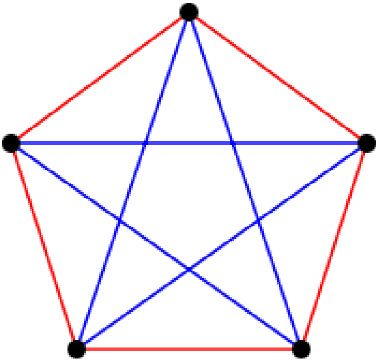
where:

- `A` - {`x | Q x`}, i.e. r(k)-short
- `B` - {`y | P y`}, i.e. BC(k)
- `R` - y "shears down" to x
- `f` - mapping from r(k)-short languages to BC(k) languages

Picture view



Ramsey theory



Ramsey on graphs

One can always find monochromatic cliques in any edge-coloring of a sufficiently large connected graph (Ramsey 1930).

Ramsey on sets

For every natural number k and finite set of colors Q, there exists a natural number $r(k)$ such that for every ordered set I with $r(k)$ elements and for every function mapping each pair (i, j) to a color $C(i, j)$, there exists a subset $J \subset I$ with k elements such that all pairs in J are mapped to the same color.

Ramsey on breakpoint sets

Theorem (Ramsey's theorem for breakpoint sets)

```
forall (k: block_pumping_constant),
exists (rk: block_pumping_constant), rk >= k /\
forall (w: word) (bps: breakpoint_set rk w)
(P: nat -> nat -> Prop),
exists (bps': breakpoint_set k w),
sublist bps' bps /\
((forall (bp1 bp2: breakpoint bps'), bp1<bp2->(P bp1 bp2))
\/ (forall (bp1 bp2: breakpoint bps'), bp1<bp2->~(P bp1 bp2)))
```

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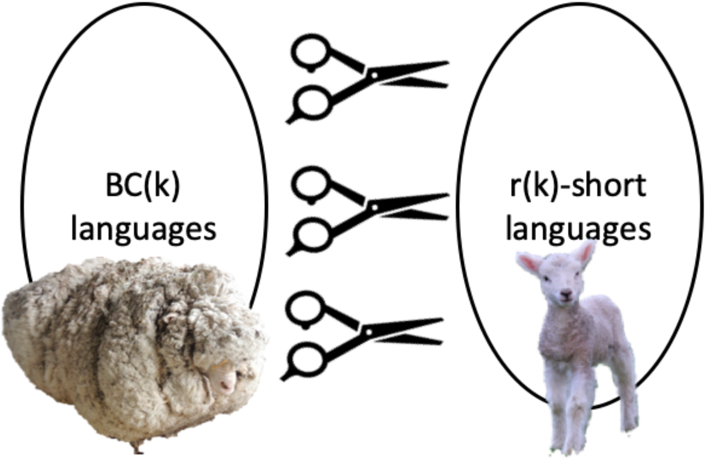
Picture view



Unshear correctness

1. Shearing an unsheared list returns us the input list;
2. Unshearing a sheared language recovers us the BC(k) language.

Unshear, pictorially

k = 3, r(k) = 6

{ 0, 1, 001, 011, 0011, 01111, 000111, ...}

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k = 3, r(k) = 6

{ 0, 1, 001, 011, 0011, 01111, 000111, ...}

0000000, 0000001, 0000010, 0000011, 0000111, 0000110, 0000101,
0001110, 0000100, 0001010, ..., 1011111, 0111111, 1111111

All words of length r(k)+1

Unshear, pictorially

k = 3, r(k) = 6

{ 0, 1, 001, 011, 0011, 01111, 000111, ...}

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All words of length r(k)+1

{ 0, 1, 001, 011, 0011, 01111, 000111, ..., 0000000, 0000001, 0000011, 0000111, 0111111, 1111111}

The chucking function

0 | 1 2 1 | 0 1 | 0 | 1 0 2 | 1 0 | 1 0 | 1 |

The chucking function

$$0 \mid \underbrace{1\;2\;1} \mid 0\;1 \mid 0 \mid 1\;0\;2 \mid 1\;0 \mid 1\;0 \mid 1 \mid$$

The chucking function

$$0 \mid \underbrace{1\ 2\ 1 \mid 0\ 1} \mid 0 \mid 1\ 0\ 2 \mid 1\ 0 \mid 1\ 0 \mid 1 \mid$$

The chucking function

$$0\underbrace{|\,1\,2\,1\,|\,0\,1\,|\,0\,|}\,1\,0\,2\,|\,1\,0\,|\,1\,0\,|\,1\,|$$

Chucking condition

Definition (Chucking function)

```
Definition chuck (k rk n : nat) (lref : list word) :=
filter (fun w => exists_all_pumps w lref k)
(generate_words_of_length (n + rk))
++ lref.
```

Definition (Chucking condition)

```
Definition chuck_prop (k rk n : nat) (lref : list word)
(w : word) :=
(exists_all_pumps_bps_prop w lref k /\ length w = n + rk)
\/ In w lref.
```

Unshear, pseudocode

1. Input: well-formed list of short words lw and candidate word w
2. Output: membership for w in $BC(k)$ language that lw agrees with on short words
3. Algorithm: incrementally consider sets of all words from length $r(k)$ to length $|w|$
    1. For every word, check the existence of a "full" set of k breakpoints such that every pair of breakpoints pumps the word down into lw
    2. If such a set exists, chuck, i.e. add, word into reference list lw
    3. If such a set does not exist, ignore
    4. Repeat with updated lw and set of words of length plus one until $|w|$ is reached
    5. Check membership of w in lw

Unshear correctness

1. Shearing an unsheared list returns us the input list;
2. Unshearing a sheared language recovers us the language.

```
Theorem unshear_correctness:
forall (k: block_pumping_constant),
exists (rk: block_pumping_constant),
forall (l: bc_language_dec k) (lw: list word),
agreement_upto k rk l (chuck_length k rk lw 0) 0 ->
(forall w, In w lw <-> (shear_language rk (unshear k rk lw)) w
/\ unshear k rk lw = (bc_language_dec_proj1 l).
```

shear (unshear)

Trivial!

unshear (shear)

`chuck` is chucking in the right words!

```
Lemma IH_chuck_step: forall (k: block_pumping_constant),
exists (rk: block_pumping_constant),
forall (l: bc_language_dec k) (lw: list word) (m: nat),
agreement_upto k rk l lw m ->
agreement_upto k rk l (chuck k rk (S m) lw) (S m).

Definition agreement_upto (k rk: block_pumping_constant)
(l: bc_language_dec k)
(lw: list word) (m : nat) :=
forall w, In w lw <-> (length w <= m + rk
/\ bc_language_dec_proj1 l w).
```

Unshear correctness

Induction step for the leftward direction:

- Given a word w that is in the new lw, we must show that:
  - $|w| \leq S\ m + k$
  - $L\ w$
- In the case that w was already in lw, we are done.
- In the case that w is newly chucked in,
  - The length requirement is satisfied by `chuck`'s specification
  - By our `chucking` condition, there exists a "full" breakpoint set lp. We specialize L's block cancelation property with w and lp to obtain a canceled word w', which agrees with w on membership.

Unshear correctness

Induction step for the rightward direction:

- Given a word w that satisfies:
  - $|w| \leq S\ m + k$
  - $L\ w$
- We must show that `chuck` actually does chuck it into lw, i.e. that it satisfies the `chuck`ing condition of having a "full" breakpoint set into lw.
- From Ramsey, we know that for any $r(k)$-size breakpoint set, we can find a monochromatic k-size breakpoint set lp.
  - In the case that all pairs in lp are cancelable pumps into lw, we satisfy the `chuck`able condition;
  - In the case that all pairs in lp are not cancelable pumps into lw, we reach a contradiction from the fact that L is $BC(k)$.

unshear (shear)

```
Lemma IH_chuck:
forall (k: block_pumping_constant),
exists (rk: block_pumping_constant),
forall (l: bc_language_dec k) (lw: list word) (m: nat),
agreement_upto k rk l (chuck_length k rk 0 lw) 0 ->
agreement_upto k rk l (chuck_length k rk m lw) m.
```

chuck is chucking in the right words!

New proof, pictorially

Lemma 2

There are finitely many BC(k) languages.

Surjectivity

unshear from well-formed short languages is surjective.

Short languages are finite

The set of r(k)-short languages is finite.

New proof, pictorially

Lemma 2

There are finitely many BC(k) languages.

Surjectivity

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Short languages are finite

The set of r(k)-short languages is finite.

Thank you!